

# Optimal control of cell mass and maturity in a model of follicular ovulation

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## Abstract

In this paper, we study optimal control problems associated with a scalar hyperbolic conservation law modeling the development of ovarian follicles. Changes in the age and maturity of follicular cells are described by a 2D conservation law, where the control terms act on the velocities. The control problem consists in optimizing the follicular cell resources so that the follicular maturity reaches a maximal value in fixed time. Using an approximation method, we prove necessary optimality conditions in the form of Pontryagin Maximum Principle. Then we derive the optimal strategy and show that there exists at least one optimal bang-bang control with one single switching time.

**Keywords:** Optimal control, conservation law, biomathematics

**2000 MR Subject Classification:** 35L65, 49J20, 92B05.

## 1 Introduction

This work is motivated by natural control problems arising in reproductive physiology. The development of ovarian follicles is a crucial process for reproduction in mammals, as its biological meaning is to free fertilizable oocyte(s) at the time of ovulation. Ovarian follicles are spheroidal, tissular structures sheltering the oocyte. During each ovarian cycle, numerous follicles are in competition for their survival. Few follicles reach an ovulatory size, since most of them undergo a degeneration process, known as atresia (see for instance [26]). The follicular cell population consists of proliferating, differentiated and apoptotic cells, and the fate of a follicle is determined by the changes occurring in its cell population in response to an hormonal control originating from the pituitary gland.

A mathematical model, using both multi-scale modeling and control theory concepts, has been designed to describe the follicle selection process on a cellular basis (see [13]). The cell population dynamics is ruled by a conservation law, which describes the changes in the distribution of cell age and maturity.

Cells are characterized by their position within or outside the cell cycle and by their sensitivity to the follicle stimulating hormone (FSH). This leads one to distinguish 3 cellular phases. Phase 1

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and 2 correspond to the proliferation phases and Phase 3 corresponds to the differentiation phase, after the cells have exited the cell cycle.

The cell population in a follicle  $f$  is represented by cell density functions  $\rho_{j,k}^f(t, x, y)$  defined on each cellular phase  $Q_{j,k}^f$ , where  $j = 1, 2, 3$  denotes Phase 1, Phase 2 and Phase 3,  $k = 1, 2, \dots$  denotes the number of the successive cell cycles (see figure 1). The cell density functions satisfy the following conservation laws:

$$\frac{\partial \rho_{j,k}^f}{\partial t} + \frac{\partial (g_f(u_f) \rho_{j,k}^f)}{\partial x} + \frac{\partial (h_f(y, u_f) \rho_{j,k}^f)}{\partial y} = -\lambda(y, U) \rho_{j,k}^f \text{ in } Q_{j,k}^f, \quad (1.1)$$

where  $Q_{j,k}^f = \Omega_{j,k}^f \times [0, T]$ , with

$$\begin{aligned} \Omega_{1,k}^f &= [(k-1)a_2, (k-1)a_2 + a_1] \times [0, y_s], \\ \Omega_{2,k}^f &= [(k-1)a_2 + a_1, ka_2] \times [0, y_s], \\ \Omega_{3,k}^f &= [(k-1)a_2, ka_2] \times [y_s, y_m]. \end{aligned}$$

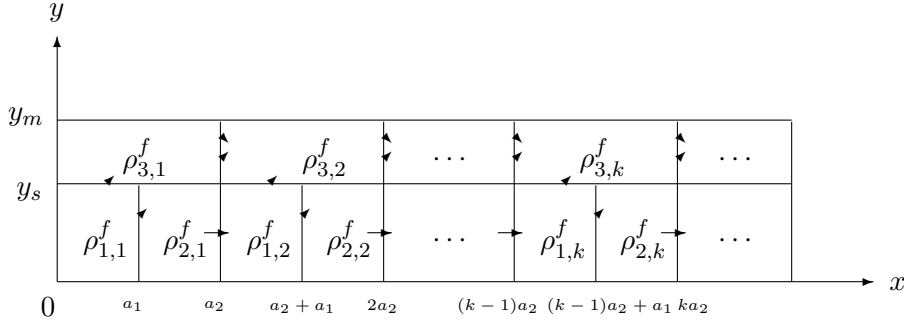


Figure 1: Cellular phases on the age-maturity plane for each follicle  $f$ . The domain consists of the sequence of  $k = 1, 2, \dots$  cell cycles. The variable  $x$  denotes the age of the cell and  $y$  denotes its maturity. The number  $y_s$  is the threshold value at which cell cycle exit occurs and  $y_m$  is the maximal maturity. The top of the domain corresponds to the differentiation phase and the bottom to the proliferation phase.

Let us define

$$M_f(t) := \sum_{j=1}^3 \sum_{k=1}^N \int_0^{+\infty} \int_0^{+\infty} y \rho_{j,k}^f(t, x, y) dx dy \quad (1.2)$$

as the maturity on the follicle scale, and

$$M(t) := \sum_f M_f(t) \quad (1.3)$$

as the maturity on the ovarian scale.

Two acting controls,  $u_f(t, M_f)$  and  $U(t, M)$  can be distinguished (see [12]). The global control  $U(t, M)$  results from the ovarian feedback onto the pituitary gland and impacts the secretion of FSH. The feedback is responsible for reducing FSH release, leading to the degeneration of all but

those follicles selected for ovulation. The local control  $u_f(t, M_f)$  is specific to each follicle and accounts for the modulation in FSH bioavailability related to follicular vascularization.

In Phase 1 and 3, both a global control  $U$  and a local control  $u_f$  act on the velocities of aging ( $g_f$  function, locally controlled in Phase 1), and maturation ( $h_f$  function, locally controlled in Phase 1 and 3) as well as on the loss term ( $\lambda$  apoptosis rate, globally controlled in Phase 1 and 3). Phase 2 is uncontrolled ( $g_f = \tau_{gf}$ ,  $h_f = \lambda = 0$ ) and corresponds to completion of mitosis after a pure delay in age  $a_2 - a_1$  (no cells leave the cycle here). The increase in the cell mass occurs through mitosis. At the time of mitosis, a mother cell gives birth to two daughter cells, which results in a local doubling of the flux. Accordingly, the boundary between different cell cycles  $k = 1, 2, \dots$  is expressed as (see figure 1)

$$g_f(u_f)\rho_{1,k+1}^f(t, ka_2, y) = 2\tau_{gf}\rho_{2,k}^f(t, ka_2, y), \quad (t, y) \in [0, T] \times [0, y_s].$$

One can refer to [29] for more details on the model.

The aging velocity controls the duration of the cell division cycle. Once the cell age has reached a critical age, the mitosis event is triggered and the two daughter cells enter a new cell cycle. Hence, there are local singularities in the subpart of the domain where  $y \leq y_s$ , that correspond to the flux doubling due to the successive mitosis events. The maturation velocity controls the time needed to reach a threshold maturity  $y_s$ , when the cell exits the division cycle definitively. After the exit time, the cell is no more able to contribute to the increase in the follicular cell mass.

Ovulation is triggered when the ovarian maturity reaches a threshold value  $M_s$ . The stopping time  $T_s$  is defined as

$$T_s := \min \{T \mid M(T) = M_s\}, \quad (1.4)$$

and corresponds on the biological ground to the triggering of a massive secretion of the hypothalamic gonadotropin releasing hormone (GnRH).

As a whole, system (1.1)-(1.3) combined with stopping condition (1.4) defines a multiscale reachability problem. It can be associated to an optimal control problem that consists in minimizing  $T_s$  for a given target maturity  $M_s$ .

The follicles are then sorted according to their individual maturity. The ovulatory follicles are those whose maturity at time  $T_s$  has overpassed a threshold  $M_{s1}$  such as  $M_{s1} < M_s$ . The ovulation rate can then be computed as

$$N_{s,s1} = \text{Card} \{f \mid M_f(T_s) \geq M_{s1}\}. \quad (1.5)$$

The hormonal control exerted by FSH acts directly on follicular cells to control their commitment towards either proliferation or differentiation. In turn, the hormonal feedback exerted by the ovary on the hypothalamo-pituitary axis ensues from the weighted contribution of all cells distributed amongst all follicles. Hence, a hormone-driven competition process, occurring within the population of simultaneously developing follicles, is intertwined with each cell dynamics process taking place within a given follicle.

A concept central to the understanding of these entangled processes is that of the management of follicular cell resources, both on the follicular (intensity of selection) and ovarian (triggering and chronology of ovulation) scales. There is indeed a finely tuned balance between the production of new cells through proliferation, that increases the whole cell mass, and the maturation of cells, that increases their contribution to hormone secretion.

This concept has already been investigated on a mathematical ground. In [12], the authors studied the characteristics associated with a follicle as an open-loop control problem. They described the sets of microscopic initial conditions compatible with either ovulation or atresia in the

framework of backwards reachable set theory. Since these sets were largely overlapping, their results illustrate the prominent impact of cell dynamics control in the model. In [27], the author focused on the issue of the selection process in a game theory approach, where one follicle plays against all the other ones. Whether the follicle becomes atretic (doomed) or ovulatory (saved) depends on the follicular cell mass reached at the time when all cells stop proliferating.

In this paper, we study an optimal control problem of follicular cell resources, where the control acts directly on the velocity terms. The controllability of nonlinear hyperbolic equations (or systems) have been widely studied for a long time; for the 1D case, see, for instance [7, 9, 11, 15, 19, 23, 24, 25, 32] for smooth solutions and [1, 3, 14, 20] for  $BV$  entropic solutions. In particular, [8] provides a comprehensive survey of controllability of partial differential equations including nonlinear hyperbolic systems. As far as optimal control problems for hyperbolic systems are concerned, one can refer to [16, 17, 18, 30]. However, most of these monographs study the case where the controls are either applied inside the domain or on the boundary. Our control problem is quite different from the problems already studied in the literature, since the control terms appear in the flux. To solve the problem, we make use both of analytical methods based on Pontryagin Maximum Principle (PMP) and numerical computations.

The paper is organized as follows. In section 2, we set the optimal control problem, together with our simplified assumptions, and we enunciate the main result. In section 3, we give some optimal results in the case where Dirac masses are used as a rough approximation of the density. We show that for finite Dirac masses, every measurable optimal control is a bang-bang control with one single switching time. In addition to the theoretical results, we give some numerical illustrations. In section 4, we go back to the original PDE formulation of the model, and we show that there exists at least one optimal bang-bang control with one single switching time.

## 2 Problem statement and introductory results

### 2.1 Optimal control problem

We consider the following conservation law on a fixed time horizon:

$$\rho_t + \rho_x + ((a(y) + b(y)u)\rho)_y = c(y)\rho, \quad t \in (t_0, t_1), \quad x > 0, \quad y > 0, \quad (2.1)$$

where

$$a(y) := -y^2, \quad b(y) := c_1 y + c_2, \quad (2.2)$$

and

$$c(y) := \begin{cases} c_s, & \text{if } y \in [0, y_s), \\ 0, & \text{if } y \in [y_s, \infty), \end{cases} \quad (2.3)$$

with  $y_s$ ,  $c_s$ ,  $c_1$  and  $c_2$  being given strictly positive constants. We assume that

$$\frac{y_s^2}{c_1 y_s + c_2} < 1. \quad (2.4)$$

Let us denote by  $w$  a positive constant such that

$$w \in \left( \frac{y_s^2}{c_1 y_s + c_2}, 1 \right). \quad (2.5)$$

From (2.2) and (2.5), we have

$$a(y) + b(y)u > 0, \quad \forall y \in [0, y_s], \quad \forall u \in [w, 1]. \quad (2.6)$$

Throughout this paper the control  $u$  is assumed to satisfy the constraint

$$u \in [w, 1]. \quad (2.7)$$

Let us define the exit time  $\hat{t}_0$  as

$$y(\hat{t}_0) = y_s, \quad (2.8)$$

where  $y(t)$  is the solution to the Cauchy problem

$$\dot{y} = a(y) + b(y)w, \quad y(t_0) = 0. \quad (2.9)$$

Let us point out that, by (2.6), there exists one and only one  $\hat{t}_0$  satisfying (2.8). Note that it is not guaranteed that the exit time  $\hat{t}_0$  occurs before the final time  $t_1$ , so that we may have  $\hat{t}_0 > t_1$ . When  $t > \hat{t}_0$ , all the cells are in Phase 3, i.e. their maturity is larger than the threshold  $y_s$ . After time  $\hat{t}_0$  the mass will not increase any more due to (2.3).

We assume that there is no outer influx, i.e.

$$\rho(t, 0, y) = \rho(t, x, 0) = 0, \quad \forall t \in (t_0, t_1), \quad x > 0, \quad y > 0. \quad (2.10)$$

The initial condition is

$$\rho(0, x, y) = \rho_0(x, y), \quad x \geq 0, \quad y \geq 0, \quad (2.11)$$

where  $\rho_0$  is a positive Borel measure on  $\mathbb{R} \times \mathbb{R}$  with a compact support included in  $[0, 1] \times [0, y_s]$ .

For any admissible control  $u \in L^\infty((t_0, t_1); [w, 1])$ , we define the cost function

$$J(u) := - \int_0^{+\infty} \int_0^{+\infty} y \, d\rho(t_1, x, y), \quad (2.12)$$

and we want to study the following optimal control problem:

$$\text{minimize } J(u) \text{ for } u \in L^\infty((t_0, t_1); [w, 1]). \quad (2.13)$$

A similar minimal time problem was investigated in a much simpler ODE framework [6], where the proliferating and differentiated cells were respectively pooled in a proliferating and a differentiated compartment. The author proved by PMP that the optimal strategy is a bang-bang control, which consists in applying permanently the minimal apoptosis rate and in switching once the cell cycle exit rate from its minimal bound to its maximal one.

In contrast, due to the fact that  $c$  is discontinuous, we cannot apply PMP directly here. The idea is to first consider optimal control problems for Dirac masses (see section 3), and then to pass to the limit to get optimal control results for the PDE case (see section 4). For “discontinuous” optimal control problems of finite dimension, one cannot derive necessary optimality conditions by applying directly the standard apparatus of the theory of extremal problems [4, 21, 28]. The first problem where the cost function was an integral functional with discontinuous integrand was dealt in [2]. Later, in [31], the author studied the case of a more general functional that includes both the discontinuous characteristic function and continuous terms. There, the author used approximation methods to prove necessary optimality conditions in the form of PMP. One of the difficulties of our problem is that both the integrand of the cost function and the dynamics are discontinuous.

The main result of this paper is the following theorem.

**Theorem 1.** *Let us assume that*

$$t_1 > \hat{t}_0, \quad (2.14)$$

$$2y_s - c_1 > 0 \quad \text{and} \quad c_s > \frac{a(y_s) + b(y_s)}{y_s}. \quad (2.15)$$

*Then, among all admissible controls  $u \in L^\infty((t_0, t_1); [w, 1])$ , there exists an optimal control  $u_*$  for the minimization problem (2.13) such that*

$$\exists t_* \in [t_0, t_1] \text{ such that } u_* = w \text{ in } (t_0, t_*) \text{ and } u_* = 1 \text{ in } (t_*, t_1). \quad (2.16)$$

## 2.2 Simplifications with respect to the original model

To make the initial problem tractable, we have made several simplifications on the model dynamics.

- $S_1$ . We consider only one developing follicle, i.e.  $f = 1$ ;
- $S_2$ . There is no loss term anymore, i.e.  $\lambda = 0$ ;
- $S_3$ . The age velocity is uncontrolled, i.e.  $g_f \equiv 1$ ;
- $S_4$ . The cell division is represented by a new gain term, i.e.  $c(y)$  defined by (2.3);
- $S_5$ . The target maturity  $M_s$  can always be reached in finite time.

Simplification ( $S_1$ ) means that, in this problem, we are specially interested in the coupling between the condition needed to trigger the ovulation, on the one hand, and the control of the follicular cell dynamics, on the other hand. We consider this interaction independently of the process of follicle selection, in the sense that we isolate the dynamics of one specific follicle, as if we could ignore the influence of the other growing follicles.

( $S_2$ ) to ( $S_4$ ) allow us to simplify the cell dynamics. In ( $S_2$ ), we neglect the cell death, which is quite natural when considering only ovulatory trajectories, while, in ( $S_3$ ), we consider that the cell age evolves as time (so that the cell duration is constant and uncontrolled). Moreover, the cell division process is distributed over ages with ( $S_4$ ), so that there is a new gain term in the model instead of the former mitosis transfer condition. Even if the way that cell proliferation is represented is less realistic (no real mitosis), this does not impact the dynamics too much as long as the average cell cycle duration is preserved.

Even if it is simplified, the problem studied here still captures the essential question of the compromise between proliferation and differentiation that characterizes terminal follicular development. On the one hand, the follicle can benefit from a strong and quick enlargement of its cell population. On the other hand, this enlargement occurs at the expense of the maturation of individual cells. This compromise was instanced here as a problem of composition of velocities. We chose as control variable the nonlocal variable  $u_f$ , which is the most downstream input representing FSH on the follicular level in the original model, since it intervenes directly in the aging and maturation velocities. A relatively high aging velocity tends to favor cell mass production while a relatively high maturation velocity tends to favor an increase in the average cell maturity.

As shown in section 2.4, assumptions ( $S_2$ ) and ( $S_5$ ) allow us to replace a minimal time criterion by a criterion that consists in maximizing the final maturity. Hence, from the initial, minimal time criterion, we have shifted, for sake of technical simplicity, to an equivalent problem where the final time is fixed and the optimality criterion is the follicular maturity at final time. On the biological ground, this means that for any chosen final time  $t_1$ , the resulting maturity at final time  $M_f(t_1)$  can

be chosen in turn as a maturity target which would be reached in minimal time at time  $t_1$ . It can be noticed that in the initial problem (1.4), there might be no optimal solution without assumption  $(S_5)$ , if the target maturity is higher than the maximal asymptotic reachable maturity.

### 2.3 Solution to Cauchy problem (2.1), (2.10) and (2.11)

In this section, we give the definition of a (weak) solution to the Cauchy problem

$$\begin{cases} \rho_t + \rho_x + ((a(y) + b(y)u)\rho)_y = c(y)\rho, & t \in (t_0, t_1), x > 0, y > 0, \\ \rho(t, 0, y) = \rho(t, x, 0) = 0, & t \in (t_0, t_1), x > 0, y > 0, \\ \rho(0, x, y) = \rho_0(x, y), & x > 0, y > 0, \end{cases} \quad (2.17)$$

where  $\rho_0 : [0, 1] \times [0, y_s] \rightarrow [0, +\infty)$  is given.

Let us denote by  $\bar{y}$  the asymptotic maturity, i.e. the positive root  $y$  of  $a(y) + b(y)u = 0$  with control  $u = 1$ . From (2.2), we have

$$\bar{y} = \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}. \quad (2.18)$$

Let  $u \in L^\infty((t_0, t_1); [w, 1])$ . Let us define the map

$$\begin{aligned} \Psi : [t_0, t_1] \times [0, y_s] \times L^\infty((t_0, t_1); [w, 1]) &\rightarrow [0, \bar{y}] \\ (t, y_0, u) &\mapsto \Psi(t, y_0, u) \end{aligned}$$

by requiring

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t, y_0, u) = a(\Psi(t, y_0, u)) + b(\Psi(t, y_0, u))u(t), \\ \Psi(t_0, y_0, u) = y_0. \end{cases} \quad (2.19)$$

Let  $\rho_0$  be a Borel measure on  $\mathbb{R} \times \mathbb{R}$  such that

$$\rho_0 \geq 0, \quad (2.20)$$

$$\text{and the support of } \rho_0 \text{ is included in } [0, 1] \times [0, y_s]. \quad (2.21)$$

Let  $K := [0, t_1 - t_0 + 1] \times [0, \bar{y}]$ . Let  $M(K)$  be the set of Borel measures on  $K$ , i.e. the set of continuous linear maps from  $C^0(K)$  into  $\mathbb{R}$ . The solution to Cauchy problem (2.17) is the function  $\rho : [t_0, t_1] \rightarrow M(K)$  such that, for every  $\varphi \in C^0(K)$ ,

$$\iint_K \varphi(\alpha, \beta) d\rho(t, \alpha, \beta) = \iint_K \varphi(x_0 + t - t_0, \Psi(t, y_0, u)) e^{\int_{t_0}^t c(\Psi(s, y_0, u)) ds} d\rho_0(x_0, y_0). \quad (2.22)$$

We take expression (2.22) as a definition. This expression is also justified by the fact that if  $\rho_0$  is a  $L^\infty$  function, one recovers the usual notion of weak solutions to Cauchy problem (2.17) studied in [8, 10, 29, 30], as well as by the characteristics method used to solve hyperbolic equations (see figure 2).

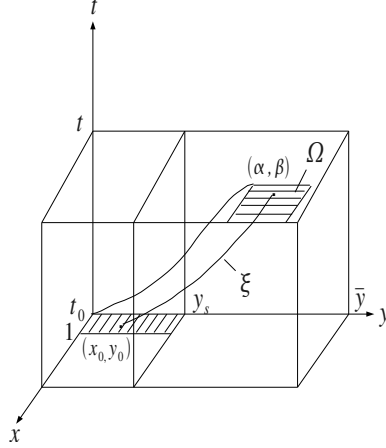


Figure 2: Construction of a weak solution by backward tracking of the characteristics. The variables  $x$  and  $y$  respectively denote the age and the maturity,  $y_s$  is the threshold maturity and  $\bar{y}$  is the asymptotic maturity. The initial mass concentrates in the shaded area  $[0, 1] \times [0, y_s]$ . The curve  $\xi = (x, y)$  is the characteristic curve passing through  $(t, \alpha, \beta)$  that intersects the initial plane  $t = t_0$  at  $(t_0, x_0, y_0)$ .

## 2.4 Minimal time versus maximal maturity

In this section, we show that the two optimal control problems enunciate either as: “minimize the time to achieve a given maturity” or “achieve a maximal maturity at a given time” are equivalent.

Let  $\rho_0$  be a nonzero Borel measure on  $\mathbb{R} \times \mathbb{R}$  satisfying (2.20) and (2.21). Let us denote by  $M^u(t)$  the maturity at time  $t$  for the control  $u \in L^\infty((t_0, t_1); [w, 1])$  (and the initial data  $\rho_0$ ).

A. For fixed target time  $t_1$ , suppose that the maximum of the maturity

$$M^u(t_1) := M \quad (2.23)$$

is achieved with an optimal control  $u \in L^\infty((t_0, t_1); [w, 1])$ . Then we conclude that for this fixed  $M$ , the minimal time needed to reach  $M$  is  $t_1$  with the same control  $u$ . We prove it by contradiction. We assume that there exists another control  $\tilde{u} \in L^\infty((t_0, \tilde{t}_1); [w, 1])$  such that

$$M^{\tilde{u}}(\tilde{t}_1) = M, \quad \tilde{t}_1 < t_1. \quad (2.24)$$

We extend  $\tilde{u}$  to  $[t_0, t_1]$  by requiring  $\tilde{u} = 1$  in  $(\tilde{t}_1, t_1]$ . Let us prove that

$$t \in [\tilde{t}_1, t_1] \rightarrow M^{\tilde{u}}(t) \text{ is strictly increasing.} \quad (2.25)$$

Let  $\tilde{\rho} : [t_0, t_1] \rightarrow M(K)$  be the solution to the Cauchy problem (see section 2.3)

$$\begin{cases} \tilde{\rho}_t + \tilde{\rho}_x + ((a(y) + b(y)u)\tilde{\rho})_y = c(y)\tilde{\rho}, & t \in (t_0, t_1), x > 0, y > 0, \\ \tilde{\rho}(t, 0, y) = \tilde{\rho}(t, x, 0) = 0, & t \in (t_0, t_1), x > 0, y > 0, \\ \tilde{\rho}(0, x, y) = \rho_0(x, y), & x > 0, y > 0. \end{cases}$$

Note that  $a(y) + b(y) > 0$  for every  $y \in [0, \bar{y})$  and that, for every  $t \in [t_0, t_1]$ , the support of  $\tilde{\rho}(t)$  is included in  $[0, t_1 - t_0 + 1] \times [0, y_s]$ . Together with (2.22) for  $\rho = \tilde{\rho}$  and  $\varphi(\alpha, \beta) = \beta$ , this proves



(2.25). From (2.25) it follows that

$$M^{\tilde{u}}(t_1) > M^{\tilde{u}}(\tilde{t}_1) = M, \quad (2.26)$$

which is a contradiction with the optimality of  $u$ .

B. For any fixed target maturity  $M$ , suppose that the minimal time needed to reach  $M$  is  $t_1$  with control  $u \in L^\infty((t_0, t_1); [w, 1])$ . Then we conclude that for this fixed target time  $t_1$ , the maximal maturity at time  $t_1$  is  $M$  with the same control  $u$ . We prove it again by contradiction. We assume that there exists another control  $\tilde{u} \in L^\infty((t_0, t_1); [w, 1])$  such that

$$M^{\tilde{u}}(t_1) > M. \quad (2.27)$$

Then by the continuity of  $M^{\tilde{u}}(t)$  with respect to time  $t$ , there exists a time  $\tilde{t}_1 < t_1$  such that

$$M^{\tilde{u}}(\tilde{t}_1) = M, \quad (2.28)$$

which is a contradiction with minimal property of  $t_1$ . This concludes the proof of the equivalence between the two optimal control problems.  $\blacksquare$

### 3 Optimal results for finite Dirac masses

In this section, we give results on the optimal control problem (2.13) when the initial data  $\rho_0 \geq 0$  is a linear combination of a finite number of Dirac masses. For  $(\alpha, \beta)^{\text{tr}} \in \mathbb{R}^2$ , we denote by  $\delta_{\alpha, \beta}$  the Dirac mass at  $(\alpha, \beta)^{\text{tr}}$ . We assume that, for some positive integer  $N$ , there exist a sequence  $((x_1^{k0}, x_2^{k0}))_{k \in \{1, \dots, N\}}$  of elements in  $[0, 1] \times [0, y_s]$  and a sequence  $(x_3^{k0})_{k \in \{1, \dots, N\}}$  of strictly positive real numbers such that

$$\rho_0 := \sum_{k=1}^N x_3^{k0} \delta_{x_1^{k0}, x_2^{k0}}. \quad (3.1)$$

We consider the following Cauchy problem:

$$\begin{cases} \dot{x}^k = f(x^k, u), & u \in L^\infty((t_0, t_1); [w, 1]), \quad t \in [t_0, t_1], \\ x^k(t_0) = x^{k0}, \end{cases} \quad (3.2)$$

where

$$f(x^k, u) = \begin{pmatrix} 1 \\ a(x_2^k) + b(x_2^k)u \\ c(x_2^k)x_3^k \end{pmatrix}, \quad x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \end{pmatrix}, \quad x^{k0} = \begin{pmatrix} x_1^{k0} \\ x_2^{k0} \\ x_3^{k0} \end{pmatrix}. \quad (3.3)$$

It is easy to check that the maximal solution to Cauchy problem (3.2) is defined on  $[t_0, t_1]$ .

One can also easily check that the solution to Cauchy problem (2.17), as defined in section 2.3, is

$$\rho(t) = \sum_{k=1}^N x_3^k(t) \delta_{x_1^k(t), x_2^k(t)}. \quad (3.4)$$

The cost function  $J$  defined in (2.12) now becomes

$$J(u) = \sum_{k=1}^N -x_2^k(t_1) x_3^k(t_1). \quad (3.5)$$

One of the goals of this section is to prove that there exists an optimal control for this optimal control problem and that, if (2.14) and (2.15) hold, every optimal control is bang-bang with only one switching time. More precisely, we prove the following two theorems.

**Theorem 2.** *The optimal control problem (2.13) has a solution, i.e., there exists  $u_* \in L^\infty((t_0, t_1); [w, 1])$  such that*

$$J(u_*) = \inf_{u \in L^\infty((t_0, t_1); [w, 1])} J(u).$$

**Theorem 3.** *Let us assume that (2.14) and (2.15) hold. Then, for every optimal control  $u_*$  for the optimal control problem (2.13), there exists  $t_* \in (t_0, t_1)$  such that*

$$u_* = w \text{ in } (t_0, t_*) \text{ and } u_* = 1 \text{ in } (t_*, t_1). \quad (3.6)$$

This section is organized as follows. In subsection 3.1 we prove Theorem 2. In subsection 3.2 we prove a PMP (Theorem 4) for our optimal control problem. In subsection 3.3 we show how to deduce Theorem 3 from Theorem 4.

### 3.1 Proof of Theorem 2

For sake of simplicity, we give the proof of Theorem 2 only in the case where  $N = 1$ . The case where  $N > 1$  can be treated similarly, except that the notations are more complicated. To simplify the notations we also delete the  $k = 1$  index.

Let  $(u^n)_{n \in \mathbb{N}} \subset L^\infty((t_0, t_1); [w, 1])$  be a minimizing sequence of the functional  $J$ :

$$\lim_{n \rightarrow \infty} J(u^n) = \inf_{u \in L^\infty((t_0, t_1); [w, 1])} J(u).$$

We have, using (3.2), (3.3) and (3.5),

$$J(u^n) = - \int_{t_0}^{t_1} (a(x_2^n) + b(x_2^n)u^n + c(x_2^n)x_2^n)x_3^n dt - x_2^0 x_3^0, \quad (3.7)$$

where  $x_2^n$  and  $x_3^n$  are solutions to the Cauchy problem

$$\dot{x}_2^n = a(x_2^n) + b(x_2^n)u^n, \quad x_2^n(t_0) = x_2^0, \quad (3.8)$$

$$\dot{x}_3^n = c(x_2^n)x_3^n, \quad x_3^n(t_0) = x_3^0. \quad (3.9)$$

Since  $(u^n)_{n \in \mathbb{N}}$  are in  $L^\infty((t_0, t_1); [w, 1])$ , there exists  $u_* \in L^\infty((t_0, t_1); [w, 1])$  and a subsequence  $(u^{n_k})_{k \in \mathbb{N}}$  such that  $u^{n_k} \xrightarrow{*} u_*$  in  $L^\infty(t_0, t_1)$  as  $k \rightarrow +\infty$ . For sake of simplicity, we still denote the subsequence by  $(u^n)_{n \in \mathbb{N}}$ .

Let  $x_2$  and  $x_3$  be the solutions to the Cauchy problem

$$\dot{x}_2 = a(x_2) + b(x_2)u_*, \quad x_2(t_0) = x_2^0, \quad (3.10)$$

$$\dot{x}_3 = c(x_2)x_3, \quad x_3(t_0) = x_3^0. \quad (3.11)$$

Similarly to (3.7), one has

$$J(u_*) = - \int_{t_0}^{t_1} (a(x_2) + b(x_2)u_* + c(x_2)x_2)x_3 dt - x_2^0 x_3^0. \quad (3.12)$$

By (2.2), (3.8) and using the property  $w \leq u^n \leq 1$ , there exists a constant  $C$  such that

$$\|x_2^n\|_{W^{1,\infty}} \leq C, \quad \forall n \in \mathbb{N}. \quad (3.13)$$

In (3.13) and until the end of the proof of Theorem 2, we denote by  $C$  different positive constants which are independent of  $n$ . It follows from the Arzelà-Ascoli theorem and (3.13) that there exists

$\tilde{x}_2 \in C^0([t_0, t_1])$  and a subsequence  $(x_2^{n_l})_{l \in \mathbb{N}}$  such that  $x_2^{n_l} \rightarrow \tilde{x}_2$  in  $C^0([t_0, t_1])$  as  $l \rightarrow +\infty$ . Now, we choose the corresponding subsequence  $(u^{n_l})_{l \in \mathbb{N}}$  and again, we write  $(u^n)_{n \in \mathbb{N}}$  instead of  $(u^{n_l})_{l \in \mathbb{N}}$  and  $(x_2^n)_{n \in \mathbb{N}}$  instead of  $(x_2^{n_l})_{l \in \mathbb{N}}$ . We have

$$u^n \xrightarrow{*} u_* \quad \text{in } L^\infty(t_0, t_1) \quad \text{as } n \rightarrow +\infty, \quad (3.14)$$

$$x_2^n \rightarrow \tilde{x}_2 \quad \text{in } C^0([t_0, t_1]) \quad \text{as } n \rightarrow +\infty. \quad (3.15)$$

By solving (3.8), we have

$$x_2^n(t) = x_2^0 + \int_{t_0}^t (a(x_2^n) + b(x_2^n)u^n)ds, \quad \forall t \in [t_0, t_1]. \quad (3.16)$$

Using (3.14) and (3.15) and letting  $n \rightarrow +\infty$  in (3.16), we obtain

$$\tilde{x}_2(t) = x_2^0 + \int_{t_0}^t (a(\tilde{x}_2) + b(\tilde{x}_2)u_*)ds, \quad \forall t \in [t_0, t_1],$$

which together with (3.10) shows that

$$\tilde{x}_2(t) = x_2(t), \quad \forall t \in [t_0, t_1]. \quad (3.17)$$

From (3.15) and (3.17), we have

$$x_2^n \rightarrow x_2 \quad \text{in } C^0([t_0, t_1]) \quad \text{as } n \rightarrow +\infty. \quad (3.18)$$

Let us only treat the case where

$$x_2(t_1) > y_s, \quad (3.19)$$

(the case  $x_2(t_1) = y_s$  can be treated similarly and the case  $x_2(t_1) < y_s$  is simpler). By (3.19), there exists  $\hat{t} \in [t_0, t_1]$  such that

$$x_2(\hat{t}) = y_s. \quad (3.20)$$

Moreover, by (2.6), this  $\hat{t}$  is unique. From (3.18) and (3.19), we have, for  $n$  large enough, which will be from now on always assumed,

$$x_2^n(t_1) > y_s. \quad (3.21)$$

Hence, as for  $x_2$ , there exists one and only one  $\hat{t}_n \in [t_0, t_1]$  such that

$$x_2^n(\hat{t}_n) = y_s. \quad (3.22)$$

Let us prove that

$$\hat{t}_n \rightarrow \hat{t} \quad \text{as } n \rightarrow +\infty. \quad (3.23)$$

In order to prove (3.23), we may assume, without loss of generality, the existence of  $\tilde{t} \in [t_0, t_1]$  such that

$$\hat{t}_n \rightarrow \tilde{t} \quad \text{as } n \rightarrow +\infty. \quad (3.24)$$

Letting  $n \rightarrow +\infty$  in (3.18), and using (3.22) together with (3.24), we get that

$$x_2(\tilde{t}) = y_s. \quad (3.25)$$

Since  $\hat{t}$  is characterized by (3.20), (3.23) follows from (3.24) and (3.25).

From (2.3), (3.9) and (3.11), we have

$$x_3^n(t) = x_3^0 e^{c_s(t-t_0)}, \forall t \in [t_0, \hat{t}_n] \text{ and } x_3^n(t) = x_3^0 e^{c_s(\hat{t}_n-t_0)}, \forall t \in [\hat{t}_n, t_1], \quad (3.26)$$

$$x_3(t) = x_3^0 e^{c_s(t-t_0)}, \forall t \in [t_0, \hat{t}] \text{ and } x_3(t) = x_3^0 e^{c_s(\hat{t}-t_0)}, \forall t \in [\hat{t}, t_1]. \quad (3.27)$$

From (3.23), (3.26) and (3.27), we get that

$$x_3^n \rightarrow x_3 \quad \text{in } C^0([t_0, t_1]) \text{ as } n \rightarrow +\infty. \quad (3.28)$$

Finally, combining (3.7), (3.12), (3.14), (3.18) and (3.28), we get

$$\lim_{n \rightarrow +\infty} J(u^n) = J(u_*),$$

which shows that  $u_*$  is an optimal control. This concludes the proof of Theorem 2.  $\blacksquare$

### 3.2 Pontryagin Maximum Principle

In this section we prove a PMP for our optimal control problem. For sake of simplicity, we denote, from now on,

$$p(x, u) := -(a(x_2) + b(x_2)u)x_3, \quad q(x) := -c_s x_2 x_3, \quad \forall x = (x_1, x_2, x_3)^{\text{tr}} \in \mathbb{R}^3. \quad (3.29)$$

Let us denote by  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  the characteristic function of  $(-\infty, y_s)$ , i.e.

$$\chi(x_2) = \begin{cases} 1, & \forall x_2 \in (-\infty, y_s), \\ 0, & \forall x_2 \in [y_s, +\infty). \end{cases} \quad (3.30)$$

Hence, the cost function  $J$  for our optimal control problem (2.13) is

$$J(u) = \int_{t_0}^{t_1} (p(x, u) + q(x) \chi(x_2)) dt - x_2^0 x_3^0. \quad (3.31)$$

Let us define the Hamiltonian

$$\begin{aligned} \mathcal{H} : (\mathbb{R}^3)^N \times \mathbb{R} \times (\mathbb{R}^3)^N &\rightarrow \mathbb{R} \\ (x, u, \psi) = ((x^1, x^2, \dots, x^N), u, (\psi^1, \psi^2, \dots, \psi^N)) &\mapsto \mathcal{H}(x, u, \psi) \end{aligned}$$

by

$$\mathcal{H}(x, u, \psi) := \sum_{k=1}^N \langle f(x^k, u), \psi^k \rangle - \sum_{k=1}^N (p(x^k, u) + q(x^k) \chi(x_2^k)). \quad (3.32)$$

In (3.32) and in the following  $\langle a, b \rangle$  denotes the usual scalar product of  $a \in \mathbb{R}^3$  and  $b \in \mathbb{R}^3$ . Let us also define the Hamilton-Pontryagin function  $H : (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N \rightarrow \mathbb{R}$  by

$$H(x, \psi) := \max_{u \in [w, 1]} \mathcal{H}(x, u, \psi). \quad (3.33)$$

Our goal in this section is to prove the following theorem.

**Theorem 4.** Let  $u_* \in L^\infty((t_0, t_1); [w, 1])$  be an optimal control for the optimal control problem (2.13). Let  $x_*^k = (x_{*1}^k, x_{*2}^k, x_{*3}^k)^{tr}$ ,  $k = 1, \dots, N$ , be the corresponding optimal trajectory, i.e.  $x_{*1}^k \in W^{1,\infty}(t_0, t_1)$ ,  $x_{*2}^k \in W^{1,\infty}(t_0, t_1)$ ,  $x_{*3}^k \in W^{1,\infty}(t_0, t_1)$  are solutions to the following Cauchy problems

$$\dot{x}_{*1}^k = 1, \quad x_{*1}^k(t_0) = x_1^{k0}, \quad (3.34)$$

$$\dot{x}_{*2}^k = a(x_{*2}^k) + b(x_{*2}^k) u_*, \quad x_{*2}^k(t_0) = x_2^{k0}, \quad (3.35)$$

$$\dot{x}_{*3}^k = c(x_{*2}^k) x_{*3}^k, \quad x_{*3}^k(t_0) = x_3^{k0}. \quad (3.36)$$

If  $y_s \in \{x_{*2}^k(t); t \in [t_0, t_1]\}$ , let  $\hat{t}_k \in [t_0, t_1]$  be the exit time for the  $k$ -th Dirac mass, i.e. the unique time  $\hat{t}_k \in [t_0, t_1]$  such that  $x_{*2}^k(\hat{t}_k) = y_s$ . If  $y_s \notin \{x_{*2}^k(t); t \in (t_0, t_1]\}$ , let  $\hat{t}_k = t_1 + 1$ . Then, there exists  $N$  vector functions  $\psi^k = (\psi_1^k, \psi_2^k, \psi_3^k)^{tr}$ , such that  $\psi_1^k \in W^{1,\infty}(t_0, t_1)$ ,  $\psi_2^k \in W^{1,\infty}((t_0, \hat{t}_k) \cup (\hat{t}_k, t_1)) \cap (t_0, t_1)$  and  $\psi_3^k \in W^{1,\infty}(t_0, t_1)$  such that

$$\dot{\psi}_1^k = 0, \quad (3.37)$$

$$\begin{aligned} \dot{\psi}_2^k = & -(a'(x_{*2}^k) + b'(x_{*2}^k) u_*) \psi_2^k - (a'(x_{*2}^k) + b'(x_{*2}^k) u_*) x_{*3}^k \\ & - c_s x_{*3}^k \chi(x_{*2}^k) \quad \text{in } ((t_0, \hat{t}_k) \cup (\hat{t}_k, t_1)) \cap (t_0, t_1), \end{aligned} \quad (3.38)$$

$$\dot{\psi}_3^k = -c_s \chi(x_{*2}^k) \psi_3^k - (a(x_{*2}^k) + b(x_{*2}^k) u_*) - c_s x_{*2}^k \chi(x_{*2}^k), \quad (3.39)$$

$$\psi_1^k(t_1) = \psi_3^k(t_1) = 0, \quad (3.40)$$

and

- if  $\hat{t}_k < t_1$ ,

$$\psi_2^k(\hat{t}_k + 0) - \psi_2^k(\hat{t}_k - 0) \in \left[ \frac{c_s x_{*3}^k(\hat{t}_k)(y_s + \psi_3^k(\hat{t}_k))}{a(y_s) + b(y_s)}, \frac{c_s x_{*3}^k(\hat{t}_k)(y_s + \psi_3^k(\hat{t}_k))}{a(y_s) + b(y_s) w} \right], \quad (3.41)$$

$$\psi_2^k(t_1) = 0, \quad (3.42)$$

- if  $\hat{t}_k = t_1$ ,

$$-\psi_2^k(t_1) \in \left[ 0, \frac{c_s x_{*3}^k(t_1) y_s}{a(y_s) + b(y_s) w} \right]. \quad (3.43)$$

Moreover the following condition holds

$$\mathcal{H}(x_*^k(t), u_*(t), \psi^k(t)) = H(x_*^k(t), \psi^k(t)) \text{ for almost every } t \in (t_0, t_1). \quad (3.44)$$

**Proof of Theorem 4.** For sake of simplicity, we give the proof only for one Dirac mass ( $N = 1$ ) and, again, we omit the index  $k = 1$ . For more than one Dirac mass, the proof is similar. Our proof is inspired from [31].

**Step 1.** Let  $(w_i)_{i \in \mathbb{N}^*}$  be a sequence of elements in  $C^\infty(\mathbb{R})$  such that

$$0 \leq w_i, \quad \int_{\mathbb{R}} w_i(x) dx = 1, \quad \text{support } w_i \subset [-1/i, 0], \quad \forall i \in \mathbb{N}^*, \quad (3.45)$$

and, for some  $C > 0$ ,

$$|w_i'(x)| \leq C i^2, \quad \forall x \in \mathbb{R}, \quad \forall i \in \mathbb{N}^*, \quad (3.46)$$

(clearly such a sequence does exist). Then, we define a sequence of functions  $(\chi_i)_{i \in \mathbb{N}^*}$  from  $\mathbb{R}$  into  $\mathbb{R}$  as follows:

$$\chi_i(x) := \int_{\mathbb{R}} \chi(y) w_i(x-y) dy = \int_{-\infty}^{y_s} w_i(x-y) dy = \int_{x-y_s}^{+\infty} w_i(z) dz, \quad \forall i \in \mathbb{N}^*, \quad \forall x \in \mathbb{R}. \quad (3.47)$$

One easily sees that the functions  $\chi_i$  thus defined possess the following properties (compare to [31, Lemma 1 and Lemma 2]):

$$0 \leq \chi_i(x_2) \leq \chi(x_2), \quad \forall i \in \mathbb{N}^*, \quad \forall x_2 \in \mathbb{R}, \quad (3.48)$$

$$\chi_i(x_2) \rightarrow \chi(x_2) \text{ as } i \rightarrow \infty, \quad \forall x_2 \in \mathbb{R}, \quad (3.49)$$

$$0 \leq \chi_i(x_2) \leq 1, \quad \forall i \in \mathbb{N}^*, \quad \forall x_2 \in \mathbb{R}, \quad (3.50)$$

$$\chi_i = 1 \text{ in } (-\infty, y_s - (1/i)] \text{ and } \chi_i = 0 \text{ in } [y_s, +\infty), \quad \forall i \in \mathbb{N}^*. \quad (3.51)$$

Let  $u_*$  be an optimal control for the optimal control problem (2.13) and let  $x_*$  be the associated trajectory. Let  $(z_i)_{i \in \mathbb{N}^*}$  be a sequence of elements of  $C^1([t_0, t_1])$  such that the following conditions hold:

$$z_i \rightarrow u_* \text{ in } L^2(t_0, t_1) \text{ as } i \rightarrow +\infty, \quad (3.52)$$

$$\sup_{t_0 \leq t \leq t_1} |z_i(t)| \leq 2, \quad i = 1, 2, \dots. \quad (3.53)$$

It is again obvious that such a sequence of functions  $(z_i)_{i \in \mathbb{N}^*}$  does exist.

Let us first prove the following lemma which will be used later.

**Lemma 5.** *There exists  $C > 0$  such that, for every  $u \in L^\infty((t_0, t_1); [w, 1])$  and every  $x_2^0 \in [0, y_s]$ , the following holds*

$$\int_{t_0}^{t_1} (\chi(x_2(t)) - \chi_i(x_2(t))) dt \leq \frac{C}{i}, \quad \forall i \in \mathbb{N}^*, \quad (3.54)$$

where  $x_2$  is the solution to the following Cauchy problem:

$$\dot{x}_2 = a(x_2) + b(x_2)u, \quad x_2(t_0) = x_2^0.$$

**Proof of Lemma 5.** The case where  $x_2(t_1) < y_s$  is trivial, we treat the case where  $x_2(t_1) > y_s$ . Then, there exists one and only one  $\hat{t} \in [t_0, t_1]$  and one and only one  $\bar{t}_i \in [t_0, t_1]$  depending on  $i$  such that

$$x_2(\hat{t}) = y_s, \quad x_2(\bar{t}_i) = \max(y_s - (1/i), x_2^0). \quad (3.55)$$

Using (3.48) to (3.51), we obtain

$$\int_{t_0}^{t_1} (\chi(x_2(t)) - \chi_i(x_2(t))) dt = \int_{\bar{t}_i}^{\hat{t}} (\chi(x_2(t)) - \chi_i(x_2(t))) dt \leq \hat{t} - \bar{t}_i. \quad (3.56)$$

By (3.55), we have

$$y_s = x_2^0 + \int_{t_0}^{\hat{t}} (a(x_2) + b(x_2)u) dt,$$

$$\max(y_s - (1/i), x_2^0) = x_2^0 + \int_{t_0}^{\bar{t}_i} (a(x_2) + b(x_2)u) dt,$$

which together give

$$\int_{\bar{t}_i}^{\hat{t}} (a(x_2) + b(x_2)u) dt \leq \frac{1}{i}. \quad (3.57)$$

By (2.6), there exists  $\sigma > 0$  such that

$$a(x_2) + b(x_2)u \geq \sigma, \quad \forall x_2 \in [0, y_s], \forall u \in [w, 1]. \quad (3.58)$$

From (3.57), by choosing  $C = \frac{1}{\sigma}$ , we get

$$0 < \hat{t} - \bar{t}_i \leq \frac{C}{i}, \quad (3.59)$$

which, together with (3.56) concludes the proof of Lemma 5. ■

Let  $f_i : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$  be defined by

$$f_i(x, u) := \begin{pmatrix} 1 \\ a(x_2) + b(x_2)u \\ c_s \chi_i(x_2) x_3 \end{pmatrix}, \quad \forall x = (x_1, x_2, x_3)^{\text{tr}} \in \mathbb{R}^3, \quad \forall u \in \mathbb{R}. \quad (3.60)$$

Let us also define  $J_i : L^\infty((t_0, t_1); [w, 1]) \rightarrow \mathbb{R}$  by

$$J_i(u) := \int_{t_0}^{t_1} (p(x, u) + q(x) \chi_i(x_2)) dt + \frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u(t) - z_i(t)|^2 dt - x_2^0 x_3^0, \quad (3.61)$$

where  $x : [t_0, t_1] \rightarrow \mathbb{R}^3$  is the solution to the Cauchy problem

$$\dot{x} = f_i(x, u), \quad x(t_0) = x^0. \quad (3.62)$$

We consider the following optimal control problem

$$\text{minimize } J_i(u) \text{ for } u \in L^\infty((t_0, t_1); [w, 1]). \quad (\mathcal{P}_i)$$

For any  $i = 1, 2, \dots$ , problem  $(\mathcal{P}_i)$  is a “smooth” optimal control problem. By a classical result in optimal control theory (see, e.g., [22, Corollary 2, p. 262]), there exists an optimal control  $u_i$  for problem  $(\mathcal{P}_i)$ . Let  $x_i$  be the optimal trajectory corresponding to the control  $u_i$  for dynamics (3.62). We have the following lemma.

**Lemma 6.** *The following holds as  $i \rightarrow +\infty$*

$$u_i \rightarrow u_* \text{ in } L^2(t_0, t_1), \quad (3.63)$$

$$x_i \rightarrow x_* \text{ in } C^0([t_0, t_1]; \mathbb{R}^3), \quad (3.64)$$

$$\chi_i(x_{i2}) \rightarrow \chi(x_{*2}) \text{ in } L^1(t_0, t_1). \quad (3.65)$$

**Proof of Lemma 6.** Since  $u_i$  is an optimal control for the optimal control problem  $(\mathcal{P}_i)$  and  $u_* \in L^\infty((t_0, t_1); [w, 1])$  is an admissible control for this problem, we have

$$J_i(u_i) \leq J_i(u_*), \quad \forall i \in \mathbb{N}^*.$$

By (3.61),

$$\begin{aligned} & \int_{t_0}^{t_1} (p(x_i(t), u_i(t)) + q(x_i(t)) \chi_i(x_{i2}(t))) dt + \frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u_i(t) - z_i(t)|^2 dt \\ & \leq \int_{t_0}^{t_1} (p(\bar{x}_*(t), u_*(t)) + q(\bar{x}_*(t)) \chi_i(\bar{x}_{*2}(t))) dt + \frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u_*(t) - z_i(t)|^2 dt, \end{aligned} \quad (3.66)$$

where  $x_i = (x_{i1}, x_{i2}, x_{i3})^{\text{tr}}$  and  $\bar{x}_* = (\bar{x}_{*1}, \bar{x}_{*2}, \bar{x}_{*3})^{\text{tr}}$  are solutions to the Cauchy problems

$$\dot{x}_i = f_i(x_i, u_i), \quad x_i(t_0) = x^0, \quad (3.67)$$

$$\dot{\bar{x}}_* = f_i(\bar{x}_*, u_*), \quad \bar{x}_*(t_0) = x^0. \quad (3.68)$$

Since  $u_*$  is an optimal control for the optimal control problem (2.13) and  $u_i$  is an admissible control for this problem, we have, for every  $i \in \mathbb{N}^*$ ,

$$\int_{t_0}^{t_1} (p(x_*(t), u_*(t)) + q(x_*(t)) \chi(x_{*2}(t))) dt \leq \int_{t_0}^{t_1} (p(\tilde{x}_i(t), u_i(t)) + q(\tilde{x}_i(t)) \chi(\tilde{x}_{i2}(t))) dt, \quad (3.69)$$

where  $x_* = (x_{*1}, x_{*2}, x_{*3})^{\text{tr}}$  and  $\tilde{x}_i = (\tilde{x}_{i1}, \tilde{x}_{i2}, \tilde{x}_{i3})^{\text{tr}}$  are solutions to the Cauchy problems

$$\dot{x}_* = f(x_*, u_*), \quad x_*(t_0) = x^0, \quad (3.70)$$

$$\dot{\tilde{x}}_i = f(\tilde{x}_i, u_i), \quad \tilde{x}_i(t_0) = x^0. \quad (3.71)$$

By (3.48) and note that  $q \leq 0$ , the inequality

$$\int_{t_0}^{t_1} q(\tilde{x}_i(t)) \chi(\tilde{x}_{i2}(t)) dt \leq \int_{t_0}^{t_1} q(\tilde{x}_i(t)) \chi_i(\tilde{x}_{i2}(t)) dt \quad (3.72)$$

holds for every  $i \in \mathbb{N}^*$ . Combining (3.69) and (3.72), we obtain

$$\begin{aligned} & \int_{t_0}^{t_1} (p(x_*(t), u_*(t)) + q(x_*(t)) \chi(x_{*2}(t))) dt \leq \\ & \int_{t_0}^{t_1} (p(\tilde{x}_i(t), u_i(t)) + q(\tilde{x}_i(t)) \chi_i(\tilde{x}_{i2}(t))) dt. \end{aligned} \quad (3.73)$$

By (3.66) and (3.73), we have

$$\begin{aligned} & \frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u_i(t) - z_i(t)|^2 dt \\ & \leq \int_{t_0}^{t_1} (p(\bar{x}_*(t), u_*(t)) + q(\bar{x}_*(t)) \chi_i(\bar{x}_{*2}(t))) dt - \int_{t_0}^{t_1} (p(x_*(t), u_*(t)) + q(x_*(t)) \chi(x_{*2}(t))) dt \\ & \quad + \int_{t_0}^{t_1} (p(\tilde{x}_i(t), u_i(t)) + q(\tilde{x}_i(t)) \chi_i(\tilde{x}_{i2}(t))) dt - \int_{t_0}^{t_1} (p(x_i(t), u_i(t)) + q(x_i(t)) \chi_i(x_{i2}(t))) dt \\ & \quad + \frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u_*(t) - z_i(t)|^2 dt. \end{aligned} \quad (3.74)$$

By definition of  $f$  and  $f_i$  in (3.3) and (3.60), we have

$$\tilde{x}_{i2}(t) = x_{i2}(t) \quad \text{and} \quad \bar{x}_{*2}(t) = x_{*2}(t), \quad \forall t \in [t_0, t_1]. \quad (3.75)$$



Using (3.29), (3.51), (3.74) and (3.75), one has

$$\begin{aligned}
\frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u_i(t) - z_i(t)|^2 dt &\leq C \int_{t_0}^{t_1} |\bar{x}_{*3}(t) - x_{*3}(t)| dt + C \int_{t_0}^{t_1} |\tilde{x}_{i3}(t) - x_{i3}(t)| dt \\
&+ C \int_{t_0}^{t_1} (\chi(x_{*2}(t)) - \chi_i(x_{*2}(t))) dt \\
&+ \frac{1}{\sqrt{i}} \int_{t_0}^{t_1} |u_*(t) - z_i(t)|^2 dt.
\end{aligned} \tag{3.76}$$

In (3.76) and until the end of the proof of Theorem 4, we denote by  $C$  different positive constants which are independent of  $i$ . From (3.48), (3.68), (3.70) and (3.75), we have

$$\begin{aligned}
\int_{t_0}^{t_1} |\bar{x}_{*3}(t) - x_{*3}(t)| dt &= \int_{t_0}^{t_1} |x_3^0 e^{\int_{t_0}^t c_s \chi_i(x_{*2}(s)) ds} - x_3^0 e^{\int_{t_0}^t c_s \chi(x_{*2}(s)) ds}| dt \\
&\leq C \int_{t_0}^{t_1} \int_{t_0}^t (\chi(x_{*2}(s)) - \chi_i(x_{*2}(s))) ds dt.
\end{aligned} \tag{3.77}$$

Applying Lemma 5 to (3.77), we get

$$\int_{t_0}^{t_1} |\bar{x}_{*3}(t) - x_{*3}(t)| dt \leq \frac{C}{i}. \tag{3.78}$$

Similarly, we can prove

$$\int_{t_0}^{t_1} |\tilde{x}_{i3}(t) - x_{i3}(t)| dt \leq \frac{C}{i}. \tag{3.79}$$

Combining (3.76), (3.78) and (3.79), and noticing (3.52), we get

$$\int_{t_0}^{t_1} |u_i(t) - z_i(t)|^2 dt \rightarrow 0 \quad \text{as } i \rightarrow +\infty. \tag{3.80}$$

From (3.52) and (3.80), we get that

$$\|u_i - u_*\|_{L^2(t_0, t_1)} \rightarrow 0 \quad \text{as } i \rightarrow +\infty. \tag{3.81}$$

Thus, property (3.63) is proved.

Let us now prove (3.64). Using in particular (3.63), one easily sees that

$$x_{i1}(t) = x_{*1}(t), \quad \forall i \in \mathbb{N}^*, \quad \forall t \in [t_0, t_1], \tag{3.82}$$

$$x_{i2} \rightarrow x_{*2} \quad \text{in } C^0([t_0, t_1]) \quad \text{as } i \rightarrow +\infty. \tag{3.83}$$

Moreover, the same proof as that of (3.28) shows that

$$x_{i3} \rightarrow x_{*3} \quad \text{in } C^0([t_0, t_1]) \quad \text{as } i \rightarrow +\infty.$$

Finally, let us prove (3.65). From (3.51) and (3.83),

$$(x_{*2}(t) \neq y_s) \Rightarrow \left( \lim_{i \rightarrow +\infty} \chi_i(x_{i2}(t)) = \chi(x_{*2}(t)) \right), \quad \forall t \in [t_0, t_1]. \tag{3.84}$$

But, as we have already seen in section 3.1, there exists at most one  $\hat{t} \in [t_0, t_1]$  such that  $x_{*2}(\hat{t}) = y_s$ . Property (3.65) follows from (3.48), (3.84) and Lebesgue's dominated convergence theorem. This concludes the proof of Lemma 6.  $\blacksquare$

**Step 2.** We now deduce necessary optimality conditions for the optimal control problem (2.13) in the form of PMP. Suppose that  $x_i$  and  $u_i$  is an optimal pair for problem  $(\mathcal{P}_i)$ . The Hamiltonian and the Hamilton-Pontryagin function for problem  $(\mathcal{P}_i)$  are respectively

$$\mathcal{H}_i(t, x, u, \psi) = \langle f_i(x, u), \psi \rangle - (p(x, u) + q(x)\chi_i(x_2)) - \frac{1}{\sqrt{i}}|u - z_i(t)|^2, \quad (3.85)$$

$$H_i(t, x, \psi) = \max_{u \in [w, 1]} \mathcal{H}_i(t, x, u, \psi). \quad (3.86)$$

By the PMP -see, e.g., [22, Theorem 2, p. 319] or [4, Section 6.5]-, there exists an absolutely continuous function  $\psi_i : [t_0, t_1] \rightarrow \mathbb{R}^3$  such that

$$\dot{\psi}_i \stackrel{a.e.}{=} - \left[ \frac{\partial f_i}{\partial x}(x_i(t), u_i(t)) \right]^{\text{tr}} \psi_i + \frac{\partial}{\partial x} (p(x_i, u) + q(x_i)\chi_i(x_{i2})), \quad (3.87)$$

$$\psi_i(t_1) = 0, \quad (3.88)$$

and

$$\mathcal{H}_i(t, x_i(t), u_i(t), \psi_i(t)) = H_i(t, x_i(t), \psi_i(t)) \text{ for almost every } t \in (t_0, t_1). \quad (3.89)$$

Let us denote  $\psi_i = (\psi_{i1}, \psi_{i2}, \psi_{i3})^{\text{tr}}$ . From (3.29), (3.60), (3.87) and (3.88), we have

$$\dot{\psi}_{i1} = 0, \quad (3.90)$$

$$\begin{aligned} \dot{\psi}_{i2} = & -(a'(x_{i2}) + b'(x_{i2})u_i) \psi_{i2} - c_s \chi'_i(x_{i2}) x_{i3} \psi_{i3} - (a'(x_{i2}) + b'(x_{i2})u_i) x_{i3} \\ & - c_s \chi_i(x_{i2}) x_{i3} - c_s x_{i2} \chi'_i(x_{i2}) x_{i3}, \end{aligned} \quad (3.91)$$

$$\dot{\psi}_{i3} = -c_s \chi_i(x_{i2}) \psi_{i3} - (a(x_{i2}) + b(x_{i2})u_i) - c_s x_{i2} \chi_i(x_{i2}), \quad (3.92)$$

$$\psi_{i1}(t_1) = \psi_{i2}(t_1) = \psi_{i3}(t_1) = 0. \quad (3.93)$$

Let  $\psi_1 : [t_0, t_1] \rightarrow \mathbb{R}$  be defined by

$$\psi_1(t) = 0, \quad \forall t \in [t_0, t_1]. \quad (3.94)$$

From (3.90), (3.93) and (3.94), one has

$$\psi_{i1}(t) = \psi_1(t), \quad \forall t \in [t_0, t_1]. \quad (3.95)$$

Let  $\psi_3 \in W^{1,\infty}(t_0, t_1)$  be the solution to the following Cauchy problem:

$$\dot{\psi}_3 = -c_s \chi(x_{*2}) \psi_3 - (a(x_{*2}) + b(x_{*2})u_*) - c_s x_{*2} \chi(x_{*2}), \quad \psi_3(t_1) = 0. \quad (3.96)$$

From (3.63), (3.64), (3.65), (3.92), (3.93) and (3.96), one easily gets that

$$\psi_{i3} \rightarrow \psi_3 \text{ in } C^0([t_0, t_1]) \text{ as } i \rightarrow +\infty. \quad (3.97)$$

We now deal with  $\psi_{i2}$ . By (3.45) and (3.47), we have

$$\text{support } \chi'_i \subset [y_s - (1/i), y_s]. \quad (3.98)$$

Moreover, from (3.45), (3.46) and (3.47), we obtain

$$|\chi'_i(x_2)| \leq Ci, \quad \forall x_2 \in [y_s - (1/i), y_s], \quad (3.99)$$

Theorem 4 in the case where  $x_{*2}(t_0) = x_2^0 = y_s$  or  $x_{*2}(t_1) < y_s$  follows directly from the standard PMP. Hence, we may assume that

$$x_{*2}(t_0) < y_s \leq x_{*2}(t_1). \quad (3.100)$$

Let us first treat the case where

$$x_{*2}(t_0) < y_s < x_{*2}(t_1). \quad (3.101)$$

Then, again, there exists one and only one  $\hat{t} \in (t_0, t_1)$  such that

$$x_{*2}(\hat{t}) = y_s. \quad (3.102)$$

Using (3.64) and (3.102), one also gets that, at least if  $i$  is large enough, which, from now on, will always be assumed, there exists one and only one  $\hat{t}_i \in (t_0, t_1)$  and one and only one  $\bar{t}_i \in (t_0, t_1)$  such that

$$x_{i2}(\hat{t}_i) = y_s, \quad x_{i2}(\bar{t}_i) = y_s - (1/i). \quad (3.103)$$

Using (3.63) and (3.64), and proceeding as in the proof of (3.23), one has

$$\hat{t}_i \rightarrow \hat{t} \text{ and } \bar{t}_i \rightarrow \hat{t} \text{ as } i \rightarrow +\infty. \quad (3.104)$$

From (3.99) and (3.103), we get

$$|\chi'_i(x_{i2}(t))| \leq Ci, \quad \forall t \in [\bar{t}_i, \hat{t}_i]. \quad (3.105)$$

From (3.98) and (3.103), we have

$$\chi'_i(x_{i2}(t)) = 0, \quad t \in [t_0, t_1] \setminus [\bar{t}_i, \hat{t}_i]. \quad (3.106)$$

Proceeding as in the proof of Lemma 5, one has

$$0 < \hat{t}_i - \bar{t}_i \leq \frac{C}{i}. \quad (3.107)$$

From (3.64), (3.91), (3.93), (3.97), (3.105), (3.106) and (3.107), we get

$$\|\dot{\psi}_{i2}\|_{L^\infty((t_0, \bar{t}_i) \cup (\hat{t}_i, t_1))} \leq C, \quad (3.108)$$

$$\|\psi_{i2}\|_{L^\infty(t_0, t_1)} \leq C. \quad (3.109)$$

From (3.108) and (3.109), and extracting if necessary a subsequence, we get the existence of  $\psi_2 \in W^{1,\infty}((t_0, \hat{t}) \cup (\hat{t}, t_1))$ , such that, as  $i \rightarrow +\infty$ ,

$$\psi_{i2} \rightarrow \psi_2 \quad \text{in } C^0([t_0, \hat{t} - \varepsilon] \cup [\hat{t} + \varepsilon, t_1]), \quad \forall \varepsilon > 0. \quad (3.110)$$

Letting  $i \rightarrow +\infty$  in (3.91), one gets, using (3.63), (3.64), (3.98), (3.106) and (3.110),

$$\begin{aligned} \dot{\psi}_2 = & - (a'(x_{*2}) + b'(x_{*2})u_*)\psi_2 - (a'(x_{*2}) + b'(x_{*2})u_*)x_{*3} \\ & - c_s x_{*3} \chi(x_{*2}) \quad \text{in } (t_0, t_1) \setminus \{\hat{t}\}. \end{aligned} \quad (3.111)$$

Moreover, from (3.93) and (3.110), we have

$$\psi_2(t_1) = 0. \quad (3.112)$$

We now prove the gap condition (3.41). Let us integrate (3.91) from  $\bar{t}_i$  to  $\hat{t}_i$ , we get

$$\psi_{i2}(\hat{t}_i) - \psi_{i2}(\bar{t}_i) = A(i) + B(i), \quad (3.113)$$

with

$$A(i) := - \int_{\bar{t}_i}^{\hat{t}_i} ((a'(x_{i2}) + b'(x_{i2}) u_i) (\psi_{i2} + x_{i3}) + c_s \chi_i(x_{i2}) x_{i3}) dt, \quad (3.114)$$

$$B(i) := - \int_{\bar{t}_i}^{\hat{t}_i} c_s x_{i3} (x_{i2} + \psi_{i3}) \chi'_i(x_{i2}) dt. \quad (3.115)$$

From (3.64), (3.104), (3.108), (3.109) and (3.114), one gets that

$$A(i) \rightarrow 0 \text{ as } i \rightarrow +\infty. \quad (3.116)$$

For  $B(i)$ , we perform the change of variable  $\tau = x_{i2}(t)$ . Using (3.103) and (3.115), we get

$$B(i) = - \int_{y_s - (1/i)}^{y_s} \frac{c_s x_{i3}(x_{i2}^{-1}(\tau)) (\tau + \psi_{i3}(x_{i2}^{-1}(\tau)))}{a(\tau) + b(\tau)u(x_{i2}^{-1}(\tau))} \chi'_i(\tau) d\tau. \quad (3.117)$$

Let us point out that, from (3.45), (3.47) and (3.51), one has

$$\int_{y_s - (1/i)}^{y_s} \chi'_i(\tau) d\tau = -1, \quad \chi'_i \leq 0. \quad (3.118)$$

From (3.58), (3.64), (3.97), (3.117), (3.118), one gets that

$$\frac{c_s x_{*3}(\hat{t})(y_s + \psi_3(\hat{t}))}{a(y_s) + b(y_s)} \leq \liminf_{i \rightarrow +\infty} B(i) \leq \limsup_{i \rightarrow +\infty} B(i) \leq \frac{c_s x_{*3}(\hat{t})(y_s + \psi_3(\hat{t}))}{a(y_s) + b(y_s)w},$$

which, together with (3.104), (3.108), (3.110), (3.113) and (3.116), gives (3.41).

Let us now treat the case where

$$x_{*2}(t_1) = y_s. \quad (3.119)$$

One adapts the proof given above in the following way.

- If, for some  $i \rightarrow +\infty$ ,  $x_{i2}(t_1) \leq y_s - (1/i)$ , then one gets  $\psi_2(t_1) = 0$ .
- If, for some  $i \rightarrow +\infty$ ,  $x_{i2}(t_1) > y_s$ , then, proceeding as above, one gets

$$-\psi_2(t_1) \in \left[ \frac{c_s x_{*3}(\hat{t})y_s}{a(y_s) + b(y_s)}, \frac{c_s x_{*3}(\hat{t})y_s}{a(y_s) + b(y_s)w} \right]. \quad (3.120)$$

- If, for some  $i \rightarrow +\infty$ ,  $y_s - (1/i) < x_{i2}(t_1) \leq y_s$ , one just need to replace  $\bar{t}_i$  by  $t_1$  in the above proof. Then, one replaces (3.118) by

$$\int_{y_s - (1/i)}^{x_{i2}(t_1)} \chi'_i(\tau) d\tau \leq -1, \quad \chi'_i \leq 0,$$

and one gets (3.120).

Let us now prove the maximum condition (3.89). By (3.63), extracting if necessary a subsequence of the  $u_i$ 's and still denoting by  $(u_i)_{i \in \mathbb{N}^*}$  the extracted sequence, there exists a Borel subset  $\mathcal{N}_0$  of  $(t_0, t_1)$  of Lebesgue measure 0 such that

$$\lim_{i \rightarrow +\infty} u_i(t) = u_*(t), \quad \forall t \in (t_0, t_1) \setminus \mathcal{N}_0. \quad (3.121)$$

Let  $i \in \mathbb{N}^*$ . By (3.89), there exists a Borel subset  $\mathcal{N}_i$  of  $(t_0, t_1)$  of Lebesgue measure 0 such that

$$\mathcal{H}_i(t, x_i(t), u_i(t), \psi_i(t)) = H_i(t, x_i(t), \psi_i(t)), \quad \forall t \in (t_0, t_1) \setminus \mathcal{N}_i. \quad (3.122)$$

Let

$$\mathcal{N} := \{\hat{t}\} \cup (\cup_{i \in \mathbb{N}^*} \mathcal{N}_i).$$

Then  $\mathcal{N}$  is a Borel subset of  $(t_0, t_1)$  of Lebesgue measure 0. Let  $u \in [w, 1]$ . By (3.122),

$$\mathcal{H}_i(t, x_i(t), u_i(t), \psi_i(t)) \geq \mathcal{H}_i(t, x_i(t), u, \psi_i(t)), \quad \forall i \in \mathbb{N}^*, \quad \forall t \in (t_0, t_1) \setminus \mathcal{N}. \quad (3.123)$$

Let  $\psi := (\psi_1, \psi_2, \psi_3)^{\text{tr}}$ . Letting  $i \rightarrow +\infty$  in (3.123) and using (3.53), (3.64), (3.84), (3.85), (3.95), (3.97), (3.110) and (3.121), one gets that

$$\mathcal{H}(x_*(t), u_*(t), \psi(t)) \geq \mathcal{H}(x_*(t), u, \psi(t)), \quad \forall t \in (t_0, t_1) \setminus \mathcal{N},$$

which gives (3.44). This concludes the proof of Theorem 4. ■

### 3.3 Proof of Theorem 3

In this section, we use the necessary optimality conditions given in Theorem 4 in order to prove Theorem 3. From now on, we assume that the target time  $t_1$  satisfies  $t_1 > \hat{t}_0$  so that all the cells will exit from Phase 1 into Phase 3 before time  $t_1$ . We give a proof of Theorem 3 in the case where  $N = 1$  in section 3.3.1. In section 3.3.3, we study the case where  $N > 1$ ; in this case we need additionally to analyze the dynamics between different exit times  $\hat{t}_k$ ,  $k = 1, 2, \dots, N$ , to obtain that there exists one and only one switching time and that the optimal switching direction is from  $u = w$  to  $u = 1$ . In both cases  $N = 1$  or  $N > 1$ , we give some numerical illustrations, respectively in section 3.3.2 and section 3.3.4.

#### 3.3.1 Proof of Theorem 3 in the case $N = 1$

The Hamiltonian (3.32) becomes

$$\begin{aligned} \mathcal{H}(x, u, \psi) &= \psi_1 + (a(x_2) + b(x_2)u)\psi_2 + c(x_2)x_3\psi_3 + (a(x_2) + b(x_2)u)x_3 + c(x_2)x_2x_3 \\ &= (a(x_2) + c(x_2)x_2)x_3 + \psi_1 + a(x_2)\psi_2 + c(x_2)x_3\psi_3 + b(x_2)(x_3 + \psi_2)u. \end{aligned} \quad (3.124)$$

Let  $u$  be an optimal control for the optimal control problem (2.13) and let  $x = (x_1, x_2, x_3)^{\text{tr}}$  be the corresponding trajectory. Note that, by (2.2),  $b(x_2) > 0$ . Then, by (3.33), (3.44) and (3.124), one has, for almost every  $t \in (t_0, t_1)$ ,

$$u(t) = 1 \quad \text{if} \quad x_3(t) + \psi_2(t) > 0, \quad (3.125)$$

$$u(t) = w \quad \text{if} \quad x_3(t) + \psi_2(t) < 0. \quad (3.126)$$

Let us recall that, under assumption (2.14) of Theorem 3, there exists one and only one  $\hat{t} \in [t_0, t_1]$  such that

$$x_2(\hat{t}) = y_s. \quad (3.127)$$

Then

$$x_2(t) > y_s, \quad \forall t \in (\hat{t}, t_1]. \quad (3.128)$$

We study the case where  $\hat{t} > t_0$ , the case  $\hat{t} = t_0$  being obvious. Thanks to (3.41), we get

$$(x_3 + \psi_2)(\hat{t} + 0) - (x_3 + \psi_2)(\hat{t} - 0) \geq c_s x_3(\hat{t}) \frac{y_s + \psi_3(\hat{t})}{a(y_s) + b(y_s)}. \quad (3.129)$$

By (3.35) and (3.39), we get

$$\frac{d(x_2 + \psi_3)}{dt} = -(x_2 + \psi_3) c_s \chi(x_2), \quad (3.130)$$

and then, using also (3.30), (3.40), (3.127) and (3.128), we obtain

$$y_s + \psi_3(\hat{t}) = (x_2 + \psi_3)(\hat{t}) = (x_2 + \psi_3)(t_1) \geq y_s. \quad (3.131)$$

Combining (3.129) with (3.131), we get

$$(x_3 + \psi_2)(\hat{t} - 0) \leq (x_3 + \psi_2)(\hat{t} + 0) - c_s x_3(\hat{t}) \frac{y_s}{a(y_s) + b(y_s)}. \quad (3.132)$$

When  $t \neq \hat{t}$ , from (2.3), (3.30), (3.36) and (3.38), we obtain

$$\frac{d(x_3 + \psi_2)}{dt} = -(a'(x_2) + b'(x_2)u)(x_3 + \psi_2). \quad (3.133)$$

Hence, using also (2.2), we have

$$(x_3 + \psi_2)(\hat{t} + 0) = (x_3 + \psi_2)(t_1) e^{-\int_{\hat{t}}^{t_1} (2x_2(s) - c_1 u(s)) ds}. \quad (3.134)$$

Using the first inequality of (2.15), (3.42), (3.128) and (3.134), we get

$$(x_3 + \psi_2)(\hat{t} + 0) \leq x_3(t_1). \quad (3.135)$$

Noticing that  $x_3(\hat{t} + 0) = x_3(t_1)$  and using (3.132) and (3.135), we get

$$(x_3 + \psi_2)(\hat{t} - 0) \leq x_3(t_1) \left(1 - c_s \frac{y_s}{a(y_s) + b(y_s)}\right). \quad (3.136)$$

From the second inequality of (2.15) and (3.136), we get

$$(x_3 + \psi_2)(\hat{t} - 0) < 0. \quad (3.137)$$

which, together with (3.133), gives us

$$(x_3 + \psi_2)(t) < 0, \quad t \in [t_0, \hat{t}). \quad (3.138)$$

Moreover, by (3.42), we have

$$(x_3 + \psi_2)(t_1) = x_3(t_1) > 0,$$

which together with (3.133), gives

$$(x_3 + \psi_2)(t) > 0, \quad t \in (\hat{t}, t_1]. \quad (3.139)$$

Taking  $t_* = \hat{t}$  and combining (3.138) and (3.139), with (3.125) and (3.126), we conclude the proof of Theorem 3 in the case where  $N = 1$ .  $\blacksquare$

### 3.3.2 Numerical illustration in the case $N = 1$

For one Dirac mass, the optimal switching time is unique. Assumption (2.15) is not necessary to guarantee that the optimal control is a bang-bang control with only one switching time. It is only used to guarantee that the optimal switching time coincides with the exit time. We give a numerical example to show that when  $c_s$  is “small”, there is no switch at all and the optimal control is constant ( $u = 1$ ), while when  $c_s$  is “large”, there is a switch occurring at the exit time (see figure 3).

The default parameter values are specified in Table 1 for the numerical studies.

$t_0$	initial time	0.0
$t_1$	final time	17.0
$c_1$	slope in the $b(y)$ function	11.892
$c_2$	origin ordinate in the $b(y)$ function	2.288
$y_s$	threshold maturity	6.0
$w$	minimal bound of the control	0.5

Table 1: Default parameter values

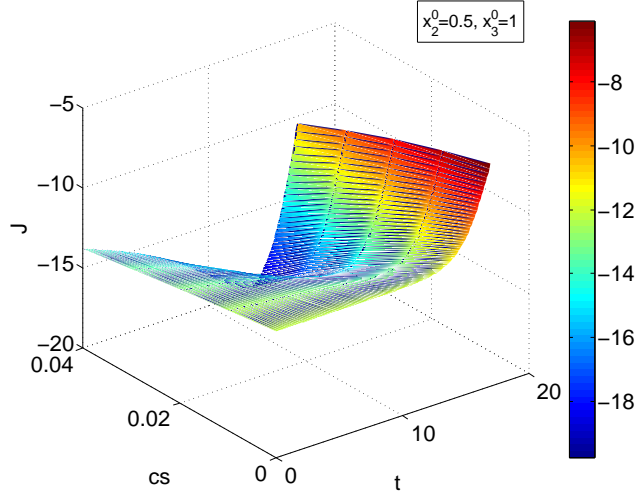


Figure 3: Value of the cost function  $J$  with respect to the switching time ( $t$ ) and  $c_s$  parameter in the case of one Dirac mass. When  $c_s$  is “small”, there is no switching time ( $t = 0$ ) and the optimal control is constant ( $u = 1$ ), while, when  $c_s$  is “large”, the optimal control strategy consists in switching from  $u = w$  to  $u = 1$  at a time coinciding with the exit time. The initial values are specified in the insert.

### 3.3.3 Proof of Theorem 3 in the case $N > 1$

Now, the Hamiltonian (3.32) becomes

$$\mathcal{H}(x, u, \psi) = \sum_{k=1}^N ((a(x_2^k) + c(x_2^k)x_2^k)x_3^k + \psi_1^k + a(x_2^k)\psi_2^k + c(x_2^k)x_3^k\psi_3^k + b(x_2^k)(x_3^k + \psi_2^k)u). \quad (3.140)$$

Reordering if necessary the  $x^k$ 's, we may assume, without loss of generality, that

$$x_2^{10} < x_2^{20} < \dots < x_2^{k0} < \dots < x_2^{N0}. \quad (3.141)$$

Let  $u$  be an optimal control for the optimal control problem (2.13) and let  $x = (x^1, \dots, x^k, \dots, x^N)$ , with  $x^k = (x_1^k, x_2^k, x_3^k)^{\text{tr}}$ , be the corresponding trajectory. From (3.141), we have

$$\hat{t}_N < \hat{t}_{N-1} < \dots < \hat{t}_k < \dots < \hat{t}_1. \quad (3.142)$$

Let  $\Phi : [t_0, t_1] \rightarrow \mathbb{R}$  be defined by

$$\Phi(t) := \sum_{k=1}^N b(x_2^k(t))(x_3^k(t) + \psi_2^k(t)). \quad (3.143)$$

Noticing that  $b(x_2^k) > 0$ , by (3.33), (3.44), (3.140) and (3.143), one has, for almost every  $t \in (t_0, t_1)$ ,

$$u = w, \quad \text{if } \Phi(t) < 0, \quad (3.144)$$

$$u = 1, \quad \text{if } \Phi(t) > 0. \quad (3.145)$$

We take the time-derivative of (3.143) when  $t \neq \hat{t}_k$ ,  $k = 1, \dots, N$ . From (2.2), we obtain

$$\dot{\Phi}(t) = \sum_{k=1}^N (c_1(x_2^k)^2 + 2c_2x_2^k)(x_3^k + \psi_2^k). \quad (3.146)$$

Similarly to the above proof for one Dirac mass, we can prove that, under assumption (2.15), we have, for each  $k = 1, \dots, N$ ,

$$(x_3^k + \psi_2^k)(t) < 0, \quad \text{when } t \in (t_0, \hat{t}_k), \quad (3.147)$$

$$(x_3^k + \psi_2^k)(t) > 0, \quad \text{when } t \in (\hat{t}_k, t_1). \quad (3.148)$$

By (3.142), (3.143), (3.147) and (3.148), and note that  $b(x_2^k) > 0$ , we get

$$\Phi(t) < 0, \quad \text{when } t \in (t_0, \hat{t}_N), \quad (3.149)$$

$$\Phi(t) > 0, \quad \text{when } t \in (\hat{t}_1, t_1). \quad (3.150)$$

The key point now is to study the dynamics of  $\Phi$  between different exit times  $\hat{t}_k$ . Let  $k \in \{1, \dots, N-1\}$  and let us assume that

$$\Phi(t) = 0, \quad \text{for some } t \in (\hat{t}_{k+1}, \hat{t}_k). \quad (3.151)$$

From (3.143) and (3.151), we get

$$x_3^k(t) + \psi_2^k(t) = - \sum_{i \neq k} \frac{b(x_2^i(t))}{b(x_2^k(t))} (x_3^i(t) + \psi_2^i(t)). \quad (3.152)$$



From (3.147) and (3.148), for every  $t \in (\hat{t}_{k+1}, \hat{t}_k)$ ,

$$x_3^i(t) + \psi_2^i(t) < 0, \quad \text{when } i \leq k-1, \quad (3.153)$$

$$x_3^i(t) + \psi_2^i(t) > 0, \quad \text{when } i \geq k+1. \quad (3.154)$$

From (2.2), (3.146) and (3.152), we get

$$\begin{aligned} \dot{\Phi}(t) = & \sum_{i \leq k-1} \frac{x_3^i + \psi_2^i}{b(x_2^k)} (c_1^2 x_2^i x_2^k + 2c_2^2 + c_1 c_2 (x_2^i + x_2^k)) (x_2^i - x_2^k) \\ & + \sum_{i \geq k+1} \frac{x_3^i + \psi_2^i}{b(x_2^k)} (c_1^2 x_2^i x_2^k + 2c_2^2 + c_1 c_2 (x_2^i + x_2^k)) (x_2^i - x_2^k). \end{aligned} \quad (3.155)$$

From (3.141), we get

$$x_2^i(t) - x_2^k(t) < 0, \quad \text{when } i \leq k-1, \quad (3.156)$$

$$x_2^i(t) - x_2^k(t) > 0, \quad \text{when } i \geq k+1. \quad (3.157)$$

Using (3.153) to (3.157), we get

$$\dot{\Phi}(t) > 0 \quad \text{whenever} \quad \Phi(t) = 0, \quad \forall t \in (\hat{t}_{k+1}, \hat{t}_k). \quad (3.158)$$

Combining (3.144), (3.145), (3.149), (3.150) and (3.158) together, we get the existence of  $t_* \in (t_0, t_1)$  such that

$$u_* = w \quad \text{in } (t_0, t_*) \quad \text{and} \quad u_* = 1 \quad \text{in } (t_*, t_1).$$

This concludes the proof of Theorem 3. ■

### 3.3.4 Numerical illustration in the case $N > 1$

The optimal control is not unique for more than one Dirac mass. Let us consider the case of two Dirac masses as an example. The optimal switching time may happen either at the first exit time or at the second exit time (see figure 4), or between the two exit times (see figure 5).

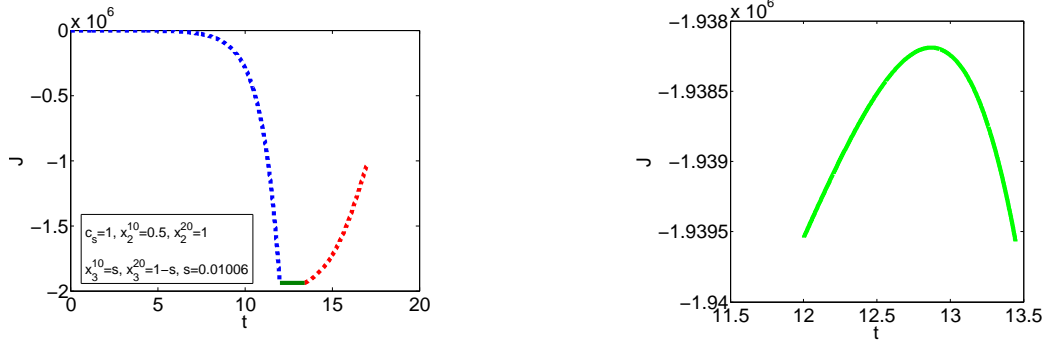


Figure 4: Value of the cost function  $J$  with respect to the switching time ( $t$ ) in the case of two Dirac masses and a “large” value of  $c_s$  ( $c_s = 1.0$ ). In the left panel, the three-part curve represents the value of the cost function obtained after switching from  $u = w$  to  $u = 1$  at time  $t$ . Blue dashed curve: switching time occurring before the first exit time; green solid curve: switching time occurring in between the two exit times; red dashed curve: switching time occurring after the second exit time. The initial values are specified in the insert. The right panel is a zoom on the green solid curve displayed on the left panel. There are two optimal switching times which coincide with the two exit times.

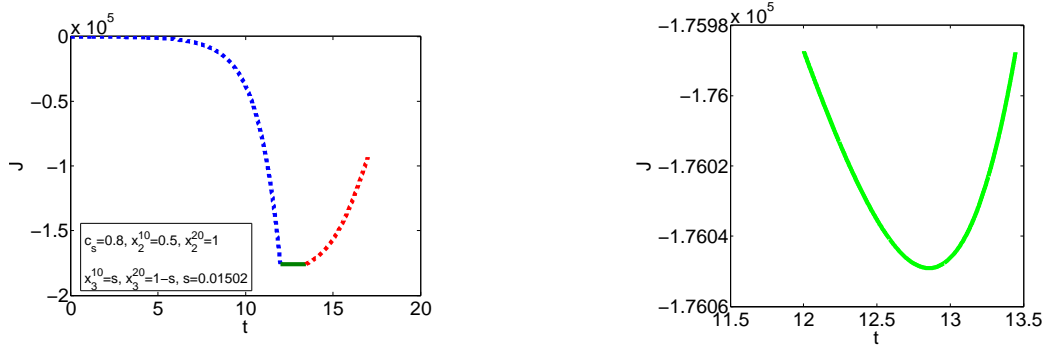


Figure 5: Value of the cost function  $J$  with respect to the switching time ( $t$ ) in the case of two Dirac masses and a “large” value of  $c_s$  ( $c_s = 0.8$ ). In the left panel, the three-part curve represents the value of the cost function obtained after switching from  $u = w$  to  $u = 1$  at time  $t$ . Blue dashed curve: switching time occurring before the first exit time; green solid curve: switching time occurring in between the two exit times; red dashed curve: switching time occurring after the second exit time. The initial values are specified in the insert. The right panel is a zoom on the green solid curve displayed on the left panel. There is one single optimal switching time, which occurs in between the two exit times.

## 4 Optimal control in the PDE case

In this section, we study the optimal control in the PDE case. We give the proof of Theorem 1. We first give an explicit expression for the cost function  $J$  defined in (2.12).

Let us define a new map

$$\begin{aligned} e : [0, y_s] \times L^\infty((t_0, t_1); [w, 1]) &\rightarrow [t_0, t_1] \\ (y_0, u) &\mapsto e(y_0, u) \end{aligned}$$

by requiring

$$\Psi(e(y_0, u), y_0, u) = y_s, \quad (4.1)$$

where  $\Psi$  is defined by (2.19). Note that, under assumption (2.14), one has, for every  $y_0 \in [0, y_s]$ , the existence of  $t \in [t_0, t_1]$  such that

$$\Psi(t, y_0, u) = y_s. \quad (4.2)$$

Again, (3.58) implies that there exists at most one  $t \in [t_0, t_1]$  such that (4.2) holds. This shows that  $e$  is well defined. Moreover, proceeding as in the proof of (3.23) and using standard results on ordinary differential equations together with (3.58) gives the following lemma.

**Lemma 7.** *Let  $(y_0^n)_{n \in \mathbb{N}}$  be a sequence of elements in  $[0, y_s]$  and  $(u^n)_{n \in \mathbb{N}}$  be a sequence of elements in  $L^\infty((t_0, t_1); [w, 1])$ . Let us assume that, for some  $y_0 \in [0, y_s]$  and for some  $u \in L^\infty((t_0, t_1); [w, 1])$ ,*

$$\begin{aligned} y_0^n &\rightarrow y_0 \text{ as } n \rightarrow +\infty, \\ u^n &\xrightarrow{*} u \text{ in } L^\infty(t_0, t_1) \text{ as } n \rightarrow +\infty. \end{aligned}$$

Then

$$e(y_0^n, u^n) \rightarrow e(y_0, u) \text{ as } n \rightarrow +\infty.$$

Let now  $\rho_0$  be a Borel measure on  $\mathbb{R} \times \mathbb{R}$  such that (2.20) and (2.21) hold. Using (2.22), (2.12) becomes

$$J(u) = - \iint_{[0,1] \times [0,y_s]} \Psi(t_1, y_0, u) e^{c_s e(y_0, u)} d\rho_0(x_0, y_0). \quad (4.3)$$

In order to emphasize the dependence of  $J$  on the initial data  $\rho_0$ , from now on we write  $J(\rho_0, u)$  for  $J(u)$ .

It is well known that there exists a sequence  $((x_0^{i,n}, y_0^{i,n}, \lambda_0^{i,n}))_{1 \leq i \leq n, n \in \mathbb{N}}$  of elements in  $[0, 1] \times [0, y_s] \times (0, +\infty)$  such that, if

$$\rho_0^n := \sum_{i=1}^n \lambda_0^{i,n} \delta_{x_0^{i,n}, y_0^{i,n}}, \quad (4.4)$$

then

$$\begin{aligned} \lim_{n \rightarrow +\infty} \iint_{[0,1] \times [0,y_s]} \varphi(x_0, y_0) d\rho_0^n(x_0, y_0) = \\ \iint_{[0,1] \times [0,y_s]} \varphi(x_0, y_0) d\rho_0(x_0, y_0), \quad \forall \varphi \in C^0([0, 1] \times [0, y_s]). \end{aligned} \quad (4.5)$$

From Theorem 2 and Theorem 3, there exists  $t_*^n \in [t_0, t_1]$  such that, if  $u_*^n : [t_0, t_1] \rightarrow [w, 1]$  is defined by

$$u_*^n = w \text{ in } [t_0, t_*^n) \text{ and } u_*^n = 1 \text{ in } (t_*^n, t_1], \quad (4.6)$$

then

$$J(\rho_0^n, u_*^n) \leq J(\rho_0^n, u), \quad \forall u \in L^\infty((t_0, t_1); [w, 1]). \quad (4.7)$$

Extracting a subsequence if necessary, we may assume without loss of generality the existence of  $t_* \in [t_0, t_1]$  such that

$$\lim_{n \rightarrow +\infty} t_*^n = t_*. \quad (4.8)$$

Let us define  $u_* : [t_0, t_1] \rightarrow [w, 1]$  by

$$u_* = w \text{ in } [t_0, t_*) \text{ and } u_* = 1 \text{ in } (t_*, t_1]. \quad (4.9)$$

Then, using (4.6), (4.8) and (4.9), one gets

$$\Psi(t_1, \cdot, u_*^n) \rightarrow \Psi(t_1, \cdot, u_*) \text{ in } C^0([0, y_s]) \text{ as } n \rightarrow +\infty. \quad (4.10)$$

Moreover, from (4.6), (4.8) and (4.9), one has

$$u_*^n \xrightarrow{*} u_* \text{ in } L^\infty(t_0, t_1) \text{ as } n \rightarrow +\infty. \quad (4.11)$$

From Lemma 7 and (4.11), one gets

$$e(\cdot, u_*^n) \rightarrow e(\cdot, u_*) \text{ in } C^0([0, y_s]) \text{ as } n \rightarrow +\infty. \quad (4.12)$$

From (4.3), (4.5), (4.10) and (4.12) and a classical theorem on the weak topology (see, e.g., [5, (iv) of Proposition 3.13, p. 63]), one has

$$J(\rho_0^n, u_*^n) \rightarrow J(\rho_0, u_*) \text{ as } n \rightarrow +\infty. \quad (4.13)$$

Let now  $u \in L^\infty((t_0, t_1); [w, 1])$ . From Lemma 7, (4.3) and (4.5), one gets

$$J(\rho_0^n, u) \rightarrow J(\rho_0, u) \text{ as } n \rightarrow +\infty. \quad (4.14)$$

Finally, letting  $n \rightarrow +\infty$  in (4.7) and using (4.13) together with (4.14), one has

$$J(\rho_0, u_*) \leq J(\rho_0, u),$$

which concludes the proof of Theorem 1. ■

## References

- [1] Fabio Ancona and Andrea Marson. On the attainable set for scalar nonlinear conservation laws with boundary control. *SIAM J. Control Optim.*, 36(1):290–312, 1998.
- [2] Aram V. Arutyunov. On necessary optimality conditions in a problem with phase constraints. *Sov. Math., Dokl.*, 31(174–177), 1985.

- [3] Alberto Bressan and Giuseppe Maria Coclite. On the boundary control of systems of conservation laws. *SIAM J. Control Optim.*, 41(2):607–622, 2002.
- [4] Alberto Bressan and Benedetto Piccoli. *Introduction to the mathematical theory of control*, volume 2 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.
- [5] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [6] Frédérique Clément. Optimal control of the cell dynamics in the granulosa of ovulatory follicles. *Math. Biosci.*, 6(123–142), 1998.
- [7] Jean-Michel Coron. Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations. *ESAIM Control Optim. Calc. Var.*, 8:513–554, 2002. A tribute to J. L. Lions.
- [8] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.
- [9] Jean-Michel Coron, Oliver Glass, and Zhiqiang Wang. Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed. *SIAM J. Control Optim.*, 48(5):3105–3122, 2009/10.
- [10] Jean-Michel Coron, Matthias Kawski, and Zhiqiang Wang. Analysis of a conservation law modeling a highly re-entrant manufacturing system. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1337–1359, 2010.
- [11] Jean-Michel Coron and Zhiqiang Wang. Controllability for a scalar conservation law with nonlocal velocity. *J. Differential Equations*, 252:181–201, 2012.
- [12] Nki Echenim, Frédérique Clément, and Michel Sorine. Multiscale modeling of follicular ovulation as a reachability problem. *Multiscale Model. Simul.*, 6(3):895–912, 2007.
- [13] Nki Echenim, Danielle Monniaux, Michel Sorine, and Frédérique Clément. Multi-scale modeling of the follicle selection process in the ovary. *Math. Biosci.*, 198(1):57–79, 2005.
- [14] Olivier Glass. On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)*, 9(3):427–486, 2007.
- [15] Martin Gugat. Boundary controllability between sub- and supercritical flow. *SIAM J. Control Optim.*, 42(3):1056–1070, 2003.
- [16] Martin Gugat. Optimal switching boundary control of a string to rest in finite time. *Z. Angew. Math. Mech.*, 88(4):283–305, 2008.
- [17] Martin Gugat, Michael Herty, Axel Klar, and Günter Leugering. Optimal control for traffic flow networks. *J. Optim. Theory Appl.*, 126(3):589–616, 2005.
- [18] Martin Gugat, Michael Herty, and Veronika Schleper. Flow control in gas networks: exact controllability to a given demand. *Math. Methods Appl. Sci.*, 34(7):745–757, 2011.

- [19] Martin Gugat and Günter Leugering. Global boundary controllability of the Saint-Venant system for sloped canals with friction. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):257–270, 2009.
- [20] Thierry Horsin. On the controllability of the Burgers equation. *ESAIM Control Optim. Calc. Var.*, 3:83–95, 1998.
- [21] Alexander D. Ioffe and Vladimir M. Tikhomirov. *Theory of extremal problems*, volume 6 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1979. Translated from the Russian by Karol Makowski.
- [22] Ernest Bruce Lee and Lawrence Markus. *Foundations of optimal control theory*. Robert E. Krieger Publishing Co. Inc., Melbourne, FL, second edition, 1986.
- [23] Tatsien Li. *Controllability and observability for quasilinear hyperbolic systems*, volume 3 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2010.
- [24] Tatsien Li and BoPeng Rao. Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM J. Control Optim.*, 41(6):1748–1755, 2003.
- [25] Tatsien Li, Bopeng Rao, and Zhiqiang Wang. Exact boundary controllability and observability for first order quasilinear hyperbolic systems with a kind of nonlocal boundary conditions. *Discrete Contin. Dyn. Syst.*, 28(1):243–257, 2010.
- [26] Elizabeth A. McGee and Aaron J. Hsueh. Initial and cyclic recruitment of ovarian follicles. *Endocr. Rev.*, 21(200–214), 2009.
- [27] Philippe Michel. Multiscale modeling of follicular ovulation as a mass and maturity dynamical system. *Multiscale Model. Simul.*, 9(1):282–313, 2011.
- [28] Lev S. Pontryagin, Vladimir G. Boltyanskii, Revaz V. Gamkrelidze, and Evgenii Frolovich Mishchenko. *The mathematical theory of optimal processes*. Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.
- [29] Peipei Shang. Cauchy problem for multiscale conservation laws: Application to structured cell populations, arxiv:1010.2132.
- [30] Peipei Shang and Zhiqiang Wang. Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system. *J. Differential Equations*, 250(2):949–982, 2011.
- [31] Alexey I. Smirnov. Necessary optimality conditions for a class of optimal control problems with a discontinuous integrand. *Tr. Mat. Inst. Steklova*, 262(Optim. Upr.):222–239, 2008.
- [32] Zhiqiang Wang. Exact controllability for nonautonomous first order quasilinear hyperbolic systems. *Chinese Ann. Math. Ser. B*, 27(6):643–656, 2006.